The use of the solution of the Geodetic Boundary Value Problem to the national linkage to an International Height Reference System: theoretical foundations and an overlook on its application to model Brazilian's IHRF stations

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Abstract

This paper addresses the theoretical foundations supporting the connection of a national vertical reference frame to an International Height Reference System (IHRS), and the main perspectives to perform such procedure in a close future for modeling Brazilian stations of the International Height Reference Frame (IHRF). It is widely known that adopting vertical systems with physical meaning is of great importance for fully representing such quantity aligned to reality; thus, in this sense and for such purpose, there is a latent need to establish a vertical reference system in a national level with a strong physical base in Brazil, fully linked to a global system with the same characteristics and according to current trends in the field. As recommended by the International Association of Geodesy (IAG), the state-of-art of altimetric determinations and vertical reference systems has, as its guideline, the search for the definition of IHRS. This definition must be made by taking as reference an equipotential surface of Earth's gravity field with pre-defined geopotential value, and its primary vertical coordinates being defined based on a geopotential difference. Meanwhile, in a Brazilian context, on July 30th, 2018, the Brazilian Institute of Geography and Statistics (IBGE) presented a readjustment of Brazilian Vertical Reference Network (BVRN) in terms of geopotential numbers and provided altimetric information on the benchmarks which compose BVRN in form of normal heights as recommended by SIRGAS Project. That step was clearly an advance towards a national alignment with the recommendations of IAG, since previously adopted representation for these benchmarks were normal-orthometric heights, i.e. heights with approximate physical meaning, produced by applying normal gravity field corrections to leveled heights. However, regardless of such advance in BVRN and some prior work needed to solve existing issues, such as a lack of connection between both Brazilian vertical data – Imbituba-SC and Santana-AP – main efforts must be directed towards accurately establishing the coordinates in the geopotential domain of the 6 (six) expected Brazilian IHRF stations, collocated with already existing stations of the Brazilian Network for Continuous Monitoring of the GNSS Systems - Fortaleza-CE (CEFT), Marabá-PA (MABA), Brasília-DF (BRAZ), Cuiabá-MT (CUIB), Presidente Prudente-SP (PPTE) and Imbituba-SC (IMBT). In turn, the calculation of geopotential values for IHRF modeling is carried out, in a modern concept, through the solution of the Geodetic Boundary Value Problem (GBVP) by different approaches, such as free and fixed scalar forms, each one with advantages and recommendations. Thus, the present paper not only presents a conceptual review on techniques for the GBVP solution, but also emphasizes its perspectives for obtaining actual IHRFates under Brazilian specific limiting factors, always aiming at the alignment with state-of-art, definitions and recommendations of IAG.

Theoretical Foundations

The primary historical objective of geodesy is to determine the shape and dimension of the Earth, as well as the gravity field outside it and its temporal variations (HOFMANN-WELLENHOF and MORITZ, 2006; TORGE and MÜLLER, 2012; GEMAEL, 2012). However, over the past decades, due to the evolution of observation techniques and computational methods employed in their processing, classical geodesy problems have been expanded and their primary function has been expanded to detect the effects of global change and geodynamics. (DREWES, 2006).

This understanding brings to light the need to understand physical quantities directly linked to the way the Earth's surface is disposed and modified over time, generating an understanding of how the Earth system behaves and bringing with it the possibility of monitoring it. Mode continuous. However, for this to be possible, not only continuous observations are necessary, but also their link to a Global Geodetic Reference System.

When it comes to altimetric information, according to IAG (2015), the search for the definition of an International Height Reference System (IHRS) should be done by reference to an equipotential surface of the gravity field. Additionally, the primary vertical coordinates must be defined based on the geopotential number C_P, expressed according to Equation 1, where $W0$ represents the geopotential value on the reference surface and WP represents the geopotential value at the calculation point.

$$C_P = -\Delta W_P = W_0 - W_P \qquad (1)$$

Currently, the geopotential calculation on a surface point can be performed by solving the Geodetic Boundary Value Problem (GBVP) according to different approaches (HOFMANN-WELLENHOF and MORITZ, 2006), such as the classical free GBVP, the scalar free GBVP, the general fixed GBVP, the linearized fixed GBVP and the simple fixed GBVP. It is state-of-art in this determination the fixed approach, with reference to the physical surface of the earth and providing independence of observation reductions and no links to local references, being the most natural approach in the age of GNSS (SÁNCHEZ, 2017; NICACIO, 2018).

Conceptual evolution of Height Reference Systems and IHRS

According to Jekeli (2000), points on or near the Earth's surface are commonly associated with three coordinates: latitude, longitude and height. The first two refer to the adopted revolution ellipsoid and are designated as geodetic latitude and longitude. For the third coordinate, however and in alignment with current propositions for height systems, it is quite plausible to take as a reference level an equipotential surface of the gravity field with fixed and known geopotential (W_0). This concept related to the height calculation, based on the difference between the equipotential surface geopotential that contains a given point P (W_P) and the reference geopotential (W_0), brings to light the concept of geopotential number C_P. This is precisely the difference between the reference geopotential W_0 and the geopotential W_P at the point (TORGE and MÜLLER, 2012), characterizing a natural measure of height that, in terms of Equation 2, is independent of the path traveled, but has no length dimension.

$$C_P = W_0 - W_P = -\int_{P_0}^{P} dW = \int g\, dn \qquad (2)$$

The use of the geopotential number in the height definition is currently the basic precept for the definition of the IHRS (IAG, 2015) and for the materialization of the associated network - the International Height Reference Frame (IHRF) (IHDE et al., 2017). From this, it is possible to adopt different hypotheses and gravity values to obtain metric units, according to Equation 3, derived from Equation 2, where H_P is the height value in a study point P, C_P is the geopotential number and g_m is an average value of gravity by some hypothesis. This equation also represents the concept of scientific height.

$$H_P = \frac{C_P}{g_m} \qquad (3)$$

According to IAG (2015), the state-of-art of geodetic observations and their applications demand the existence of a geodetic reference system with lasting stability and homogeneity throughout the Earth's surface, together with a sufficiently precise materialization to determine the magnitude of the observed achievements - on the order of a few millimeters. Thus a modern height reference system should be able to detect, for example, a millimeter order change in sea level.

In addition, Sánchez et al. (2015) point out that current height systems present a series of problems, such as: more than 100 materializations around the world; discrepancies of the order of decimeter or meter, due to different vertical *data*, different physical heights and nonexistent standardization; consideration of static heights; imprecise combination of geometric heights; and one to two orders of precision lower than required.

To this purpose, IAG (2015) determined the adoption of the following conventions for the establishment of the IHRS consonant Ihde et al. (2015):

- The vertical reference level is an equipotential surface of the earth's gravity field with geopotential value W_0;
- Parameters, observations and data must be related to the mean tide system;
- The unit of length is the meter and the unit of time is the second (SI);
- The vertical coordinates are the differences $-\Delta W_P$ between the potential of the terrestrial gravity field W_P, in points P, and the geopotential value in the geoid W_0; the potential difference $-\Delta W_P$ is also referred to as geopotential number C_P;
- The spatial reference of position P for the potential $W_P = W(\vec{X})$ is related to the $\vec{X}$ coordinates of the ITRS - International Terrestrial Reference System.

In addition to these standardizations, the same reference determined the materialization of the geopotential value on the reference surface for the IHRS as $W_0 = 62636853.4\ m^2s^{-2}$.

Height Reference Frames and IHRF

As described by Mueller (1985), the purpose of a reference frame is to provide means of materializing a reference system for a quantitative description of positions and movements. In the case of vertical reference systems, the networks associated with them are nothing but materializations or physical realizations of their vertical coordinates.

According to Sánchez et al. (2015) and Sánchez (2016), the main difficulty in establishing an IHRS realization, called IHRF, lies in the fact that currently the requirements for GGOS may not be satisfied - a global geodetic reference network with millimeter accuracy, lasting stability and homogeneity, removal of inconsistencies related to terrestrial geometry and its field of gravity, and the drafting of patterns for consistent definition and realization. This is because an IHRS materialization would be similar to an ITRS materialization, i.e., through a global network with precise and continuously monitored vertical coordinates, supported by national and regional densifications, which would imply integration and transformation between existing height systems.

Under a modern point of view, Ihde et al. (2017) define the main conventions for the realization of IHRF as:

- The reference value of W_0 is obtained from better estimates. The procedure for determining W_0 must be documented in conventions and guidelines to ensure reproduction and interpretation of changes; the value obtained by IAG (2015) is accepted;
- The central element of the IHRF is a Global Geopotential Model - GGM; This is because the availability of high resolution GGMs enables the direct calculation of $W(P)$ by introducing the ITRF $\vec{X}$ coordinates of any point into the expansion equations in spherical harmonics; according to Rummel et al. (2014 apud IHDE et al., 2017), the expected average accuracy after applying one of these models is in the order of ± 4 cm to ± 6 cm in well surveyed regions, and in the order of ± 20 cm to ± 40 cm, with extreme ± 4 m cases, in sparsely surveyed regions;
- The potential difference $-\Delta W_P$ from the agreed value W_0 should be known through a network of higher precision geodetic observation stations, where observations can be generated to derive the defining elements at the highest possible quality level, consistent with other reference systems and networks;
- The IHRF reference network should follow the same hierarchy as the ITRF reference network, i.e., a global network with national and regional densifications.

The Geodetic Boundary Value Problem (GBVP)

As presented by Carrión (2017), the boundary value problems of the potential theory applied to physical geodesy are used to determine the gravitational potential V, considered as a harmonic function. In this context, the GBVP may be, such as in the potential theory, divided into three strands, being Dirichlet's GBVP adopted in the Molodensky theory.

Dirichlet's GBVP is adopted considering as contour surface the physical surface (PS) of the Earth and making the geopotential determination at a point P of such surface, as $W_P = U_Q$(see Figure 1). In this particular case, U_Q is the spheropotential of the reference ellipsoid at point Q, which lies on the same normal line to the level ellipsoid passing through point P. For a set of points P_i on the physical surface, there will therefore be corresponding points Q_i. The set of points Q_i generates a surface called *teluroid*; the normal distance from a point on the teluroid to the corresponding P_i in PS is defined as *height anomaly* ζ_i. This same height anomaly measured over the normal since the reference ellipsoid generates the so-called *quasi-geoid*. The fundamental quantity involved in the solution of such GBVP is the so-called Molodensky gravity anomaly (or surface anomaly) given by $\Delta g_M = g_P - \gamma_Q$.

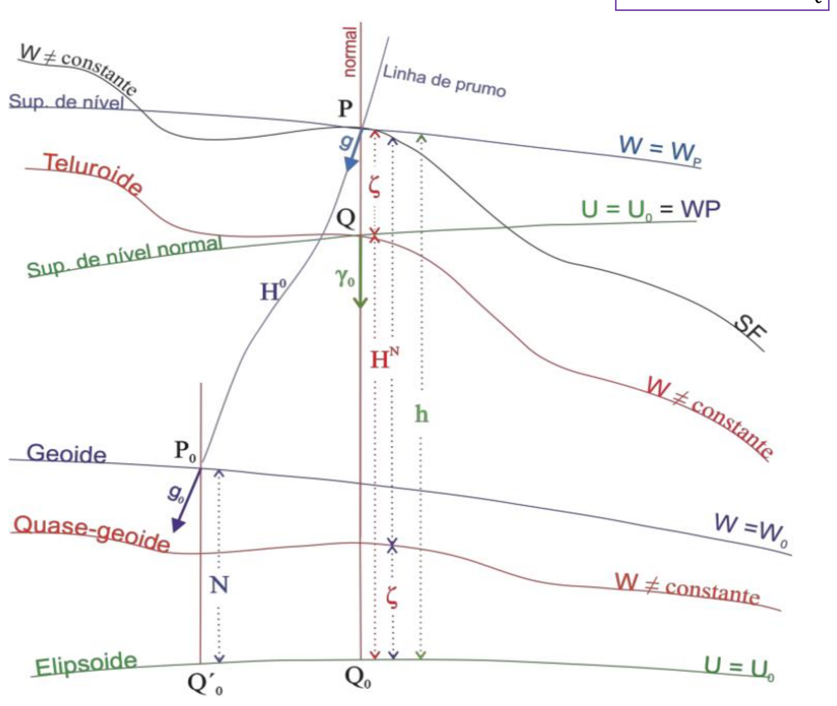

Fig 1. Reference lines and surfaces concerning GBVP

Dirichlet's GBVP is mathematically described by Equation 4, where T is known the disturbing potential:

$$\begin{cases} \Delta T = 0, & if\ r > R \\ T = \bar{T}, & if\ r = R \end{cases} \qquad (4)$$

GBVP solution approaches

As previously mentioned, GBVP might be solved according to different approaches (HOFMANN-WELLENHOF and MORITZ, 2006), such as the classical free GBVP, the scalar free GBVP, the general fixed GBVP, the linearized fixed GBVP and the simple fixed GBVP. We present ahead some different outlooks on these approaches, in line with com Hofmann-Wellenhof and Moritz (2006), Guimarães and Blitzkow (2011) and Sansò and Sideris (2013).

The Scalar Free GBVP

Scalar free GBVP may be used when geodetic coordinates (φ, λ) of the calculation point are known for every calculation point $P \in PS$. The difference between this problem and the classical free problem, for example, is that in this case the planimetric coordinates of a given point P are known about the normal to the teluroid, being necessary to know $h(\varphi, \lambda)$ to model the unknown surface. Moreover, to compensate the lack of information, two quantities of the gravity field are measured: W and g, while in the other, nothing is known about position P, and four quantities are measured: W, g, ϕ and Λ. More specific conditions for scalar free GBVP are mentioned by Heck (1989) and Sacerdote and Sansò (1986).

Figure 2 presents a synthesis of the steps used for GBVP free scalar solution based on the remove-restore technique (FORSBERG, 1997) and is, until today, used in related works, as in Palmeiro (2013), Carrión (2017) and Nicacio et al. (2018). In such figure, there is a large number of procedures and principles – such as the development of height anomaly as a geopotential functional in spherical harmonics, the principle of spectral decomposition and modeling of residual topography effects - which is not discussed in this paper. Further details on these points can be found in Ferreira (2011), Carrión (2017), Nicacio (2018) and Nicacio et al. (2018).

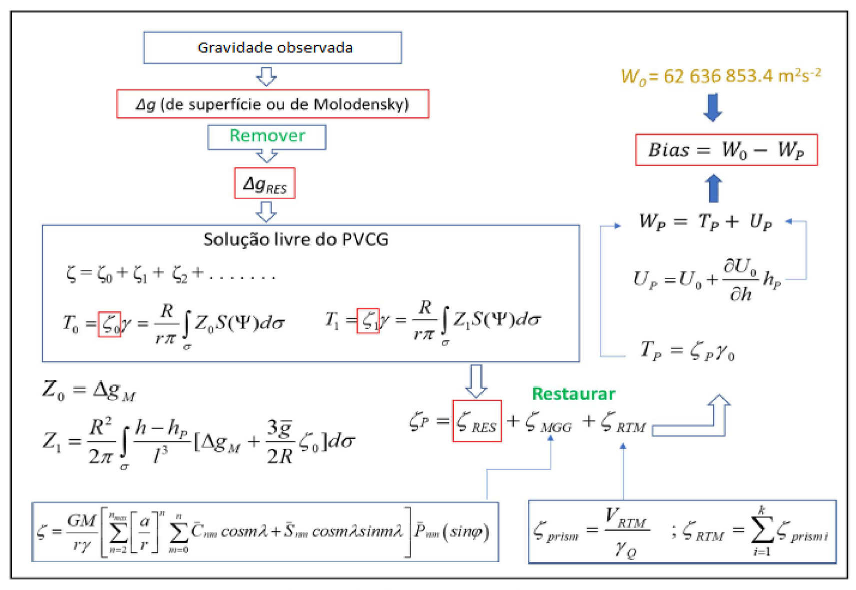

Fig 2. Scalar free GBVP solution schema

The Fixed GBVP

According to Heck (1989) and Ferreira (2011), once noted current high precision positioning technologies by artificial satellites, the fixed GBVP, assuming that the geometry of the contour surface is completely known, is increasingly practical. Considering that the terrestrial surface S is known, the unknown part of the problem is the geopotential W itself; if the gravity vector module $|\vec{g}|$ is used as a boundary condition, then the GBVP is named fixed. Figure 3 presents a synthesis of the steps used for fixed solution of GBVP also based on the remove-restore technique. Again, further details on technical procedures are omitted and may be found in references.

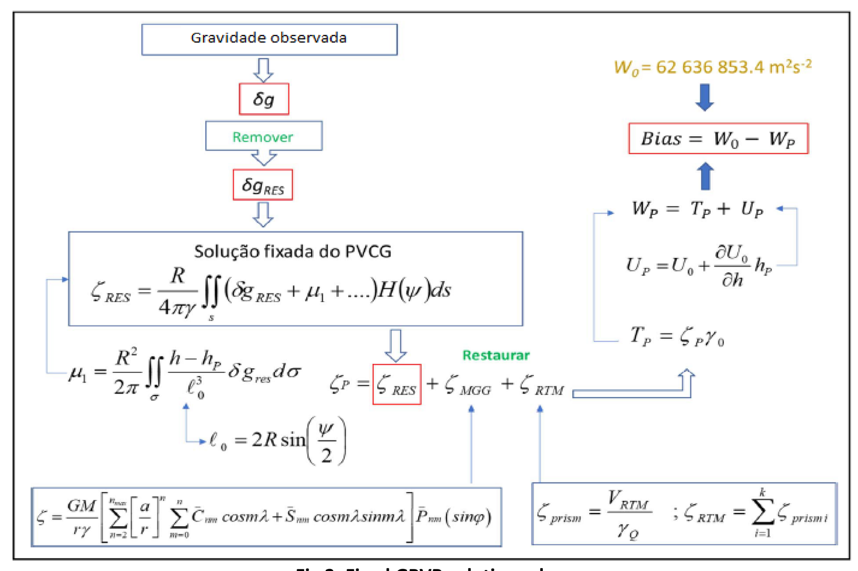

Fig 3. Fixed GBVP solution schema

The GBVP solution to a linkage to IHRS/IHRF

The integration between the well existing altimetric *data* in each country, already stablished by the competent agencies, aiming at the adaptation to the new height modeling and the elimination of discrepancies between the national height *data* and the global height *datum*, can be accomplished according to the formulation presented by Equation 5, as described by Nicacio (2017), expressed as a function of the difference between the geopotential numbers of a calculation point *P* obtained from the global datum (C_P) and the national datum (C_{P_i}). In such equation, T_P is the anomalous potential determined by the GBVP solution.

$$C_P - C_{P_i} = W_0 - W_{0_i} = \delta W_i \cong [W_0 - (U_P + T_P)] - \sum_j g_{m_j}\Delta n_j \qquad (5)$$

Figure 4 illustrates the procedure of employing the GBVP solution to link to IHRF.

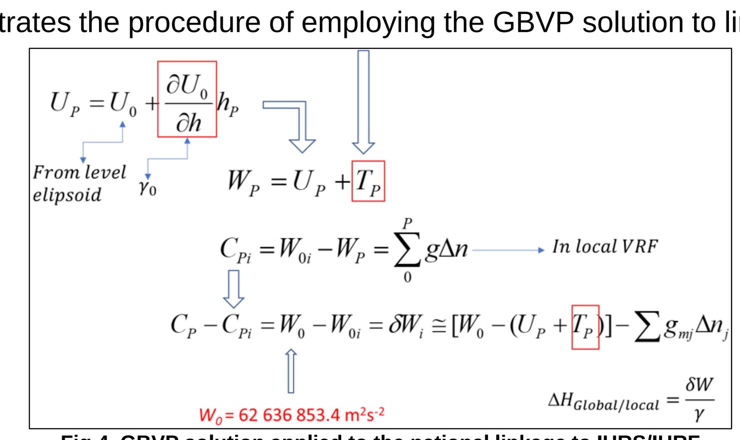

Fig 4. GBVP solution applied to the national linkage to IHRS/IHRF

Overlook on Brazilian's IHRF stations proposal

Regarding the forecast and proposition of IHRF stations on the globe, Sánchez et al. (2017) report that in April 2017,a first proposal was made with 163 possible locations of stations, following discussions with experts at regional and national levels. Regarding the Brazilian advances for materialization of IHRF stations, the current proposal for locations for IHRF stations on national land is: given the requirements presented for the installation of these stations (SÁNCHEZ et al., 2017), IBGE (Brazilian Institute of Geography and Statistics) listed 6 RBMC (Brazilian Continuous Monitoring Network) stations [refer to figure 5] located in the cities of: Fortaleza-CE (CEFT station), Marabá-PA (MABA station), Brasilia-DF (BRAZ station), Cuiabá-MT (CUIB station), Presidente Prudente-SP (PPTE station) and Imbituba-SC (IMBT station) that shall be used as future IHRF stations.

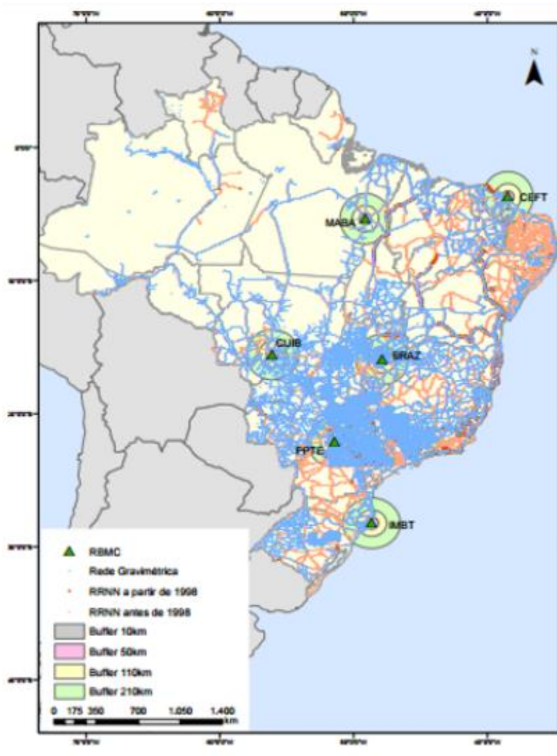

Fig 5. Proposed location of Brazilian's IHRF stations

Brazilian Vertical Reference Network (BVRN) towards IHRF

Until mid-2018, BVRN heights, called normal-orthometric heights, available in the IBGE Geodetic Database (BDG/IBGE), were derived from level differences corrected by the systematic effect caused by the non-parallelism of the equipotential surfaces of the normal gravity field, mainly caused by the lack of combination of gravity observations with leveling. In July 2018, however, the official launch of the readjusted BVRN/IBGE in terms of geopotential numbers (IBGE, 2018) was held. By combining gravimetric information with observations of "pure" level differences, it was proceeded to the adoption of normal heights as a function of theoretical gravity values in terms of Equation 3.

The calculations and analyzes developed during the network reprocessing are characterized by robustness, involving a larger number of qualitative variables in relation to the previous procedures. The insertion of new parameters provided the calculation of heights with greater physical significance and prepared the network for future actions recommended by SIRGAS on the unification of the altimetric system of the Americas (IBGE, 2018). As reinforced by IBGE (2018), the new normal heights remain referenced to the altimetric references currently in force in Brazil, Imbituba and Santana, both defined, in each case, from a mean sea level value.

Conclusions and outlook

Considering what is discussed in this paper and, with close sight in Equation 5, it can be seen that the improvements imposed by the new adjustment of the BVRN impact positively the final part of that equation; although, alternatives should be sought to act in its first part, directly by the GBVP solution - through any of the approaches presented here, for example - to calculate the anomalous potential T_P at surface points where IHRF stations are to be established and, finally, to define the local/national geopotential reference value to be linked to the global reference geopotential by the approach presented in figure 4.

References

To check the references (quotes and figures), please refer to the paper linked by the following QR code:

Acknowledgments

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